Asymptotic Performances of Multiple Turbo Codes using the Gaussian Approximation

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1 Introduction

In his thesis, Wiberg [1] has observed that if the MAP decoder inputs are gaussian and independent, then its outputs can also be approximated as gaussian random variables. Recently, El Gamal [2] has proposed to use this gaussian approximation to study the asymptotic performances of the iterative decoder of different families of concatenated codes (LDPC, turbo codes [3], ...). From this approximation, it is possible to find the minimum ratio E_b/N_{0S} corresponding to the limit from which the signal to noise ratio of the extrinsic information $SNR_{EXTR} \rightarrow \infty$ (and the bit error ratio $P_e \rightarrow 0$) when the number of iterations $m \rightarrow \infty$. This method can be used to choose the best constituent codes of parallel concatenated convolutional codes (PCC codes) from the point of view of the iterative decoder.

In this paper, we propose to generalize this method to PCC codes with more than 2 constituent codes or multiple parallel concatenated convolutional codes (MPCC codes).

We will first review the work of El Gamal on gaussian approximation [2]. Then, we will apply the gaussian approximation to different iterative decoder structures for Multiple Turbo codes. Finally, we will give some lists of thresholds for MPCC codes of rate $1/2$ with J convolutional codes of rate $\frac{J}{J+1}$. These concatenated codes are an interesting alternative to Turbo Codes. The decoding of these codes can be simpler than the classical ones.

2 Mathematical model

Let's consider that the zero codeword has been transmitted over an AWGN channel. The signal to noise ratio of a gaussian random variable x is defined by $SNR(x) = [E(x)]^2/\mathrm{VAR}(x)$ where $\mathrm{E}(x)$ and $\mathrm{VAR}(x)$ are respectively the mean and variance of x. If the information are gaussian logarithm likelihood ratios (LLR), we have the following relation: $\mathrm{VAR}(x) = 2\times\mathrm{E}(x)$. The gaussian approximation means that the decoder outputs are completely determined from its inputs. We have the relation $P_e = Q(\sqrt{SNR})$ between the bit error ratio P_e and the signal to noise ratio SNR where $Q()$ is the error function. We will consider that the information sequences are long enough to consider that the cycles in the graph have no effect on the message passing all along the iterations. In that case, we can consider that the input information $APRILLR$ et $INTLLR$ and the output information $APPLLR$ and $EXTRLLR$ of the MAP decoder are gaussian and independent. Since $1/n$ convolutional codes are isotropic [4], the signal to noise ratio of the extrinsic information SNR_{EXTR} is independent of the position of the bit in the sequence. As a consequence, we can consider that the MAP decoder is a signal to noise amplifier. In this paper, except for the parallel decoder, an iteration is associated to the decoding of one constituent code. For a given ratio E_b/N_0 of the channel information, and according to the gaussian approximation and independence hypothesis, $SNR_{EXTR}^{(m)}$ evolves at each iteration m as follows :

$$SNR_{EXTR}^{(m)} = f_{E_b/N_0}(SNR_{APRI}^{(m)}) = f_{E_b/N_0}(SNR_{EXTR}^{(m-1)}) \quad (1)$$

The function f_{E_b/N_0} gives the relation between the signal to noise ratios SNR_{APRI} and SNR_{EXTR} of the MAP decoder for a given ratio E_b/N_0. As stated hereabove, for rate $1/n$ convolutional codes, the statistical properties of the output information are independent of the bit position. For k/n convolutional codes, since these codes are anisotropic of degree $d \leq k$, we have d different functions f_{E_b/N_0}. In that case, we can generally use the mean of these functions.

Using the Lebesgue-Borel theorem, we can say that the sequence $\{SNR_{EXTR}\}_{m=0}^{\infty}$ reaches a fix point $\tau(E_b/N_0) < \infty$ or tends to ∞. Since this sequence is a continuous increasing sequence, we have $\tau(E_b/N_0) < \infty$ for $E_b/N_0 < E_b/N_{0S}$ ($P_e \neq 0 \quad \forall m$) and $\tau(E_b/N_0) = \infty$ for $E_b/N_0 > E_b/N_{0S}$ ($P_e \to 0$ when $m \to \infty$). Therefore, the signal to noise ratio E_b/N_{0S} is the threshold which determines the convergence or non convergence of the iterative decoder. The function $SNR_{EXTR} = f_{E_b/N_0}(SNR_{APRI})$ for different values of the ratio E_b/N_0 is determined by means of Monte-Carlo simulations. For Turbo codes, the threshold E_b/N_{0S} corresponds to the ratio E_b/N_0 for which the function $SNR_{EXTR} = f_{E_b/N_0}(SNR_{APRI})$ is tangent to the straight line $SNR_{EXTR} = SNR_{APRI}$.

3 Generalization to multiple parallel concatenated convolutional codes

Parallel concatenated convolutional codes with more than 2 constituent codes or multiple parallel concatenated convolutional codes (MPCC codes) have been introduced in [5]. In comparison with classical Turbo codes, different iterative decoder structures are possible when the number of constituent codes J is greater than 2 : the serial decoder, the modified serial decoder and the parallel decoder.

The figure 1 gives a simplified scheme of a serial decoder.

According to the independence hypothesis, the interleavers are not drawn in the figures. For this

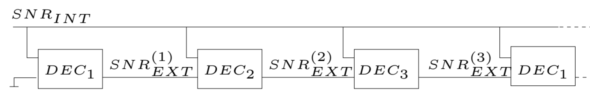

Figure 1: serial decoder

structure, the ratio $SNR_{EXTR}^{(m)}$ evolves at each iteration m as in the classical Turbo codes :

$$SNR_{EXTR}^{(m)} = f_{E_b/N_0}(SNR_{APRI}^{(m)}) = f_{E_b/N_0}(SNR_{EXTR}^{(m-1)}) \quad (2)$$

Since we only use the extrinsic information from the previous decoder for the calculation of $EXTR^{(m)}$, this decoder gives the worst asymptotic performances.

The modified serial decoder is given in figure 2.

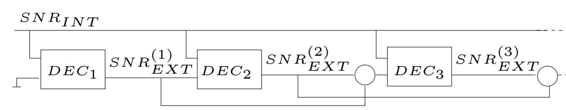

Figure 2: modified serial decoder

The *a priori* information is the sum of the extrinsic information calculated previously by the other decoders. For exemple, let's consider the case $J = 3$, the ratio $SNR_{EXTR}^{(m)}$ evolves at each iteration m as follows :

$$SNR_{EXTR}^{(m)} = f_{E_b/N_0}(SNR_{APRI}^{(m)}) = f_{E_b/N_0}(SNR_{EXTR}^{(m-1)} + SNR_{EXTR}^{(m-2)}) \quad (3)$$

In the parallel decoder given in figure 3, the J MAP decoders calculate in parallel the extrinsic information from those calculated during the previous iteration. In this case, an iteration will correspond to the decoding of J codes in parallel. With this decoder, the sequence $SNR_{EXTR}^{(m)}$ is given by :

$$SNR_{EXTR}^{(m)} = f_{E_b/N_0}(SNR_{APRI}^{(m)}) = f_{E_b/N_0}((J-1) \times SNR_{EXTR}^{(m-1)}) \quad (4)$$

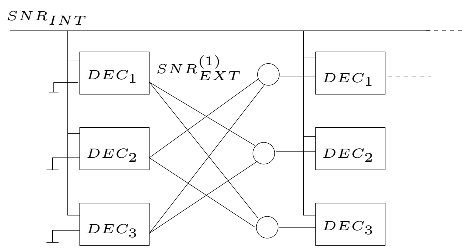

Figure 3: parallel decoder

With this structure, the threshold E_b/N_{0S} corresponds to the ratio E_b/N_0 for which the function $SNR_{EXTR} = f_{E_b/N_0}(SNR_{APRI})$ is tangent to the straight line $SNR_{EXTR} = \frac{1}{J-1} \times SNR_{APRI}$.

For modified serial and parallel decoder, there are $J-1$ of extrinsic information involved in the calculation of $EXTR^{(m)}$. As in section 2, we can use the relation (4) to calculate the threshold of the iterative decoder.

We have compared the ratio SNR_{EXTR} evolution in function of the number of iterations for the modified serial and parallel decoder on figures 4 and 5. It is clear that the ratio SNR_{EXTR} increase faster using a parallel decoder. However, the threshold E_b/N_{0S} is the same for both structures.

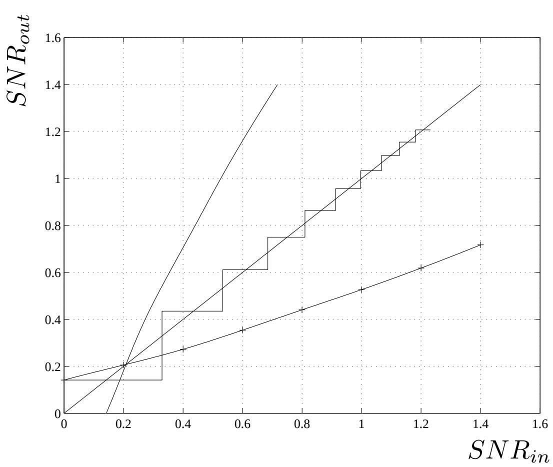

Figure 4: Evolution of the ratio SNR_{EXTR} with E_b/N_0=0.8dB for a modified serial decoder.

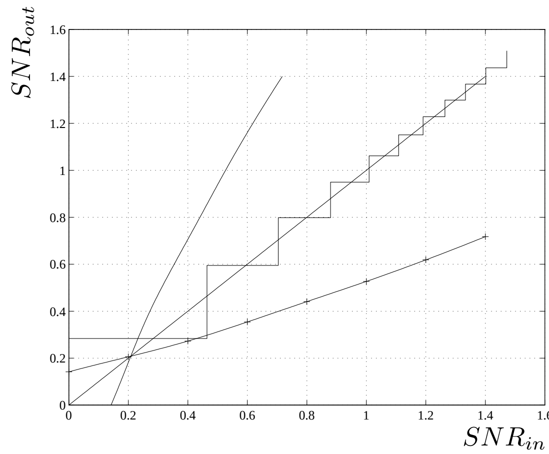

Figure 5: Evolution of the ratio SNR_{EXTR} with E_b/N_0=0.8dB for a parallel decoder.

4 Results

Tables 1, 2 and 3 give different results obtained for MPCC codes of rate 1/2 with J constituent codes of rate $\frac{J}{J+1}$. Table 1 gives the thresholds E_b/N_{0S} for different codes obtained with an exhaustive search by Benedetto *et al.*[6].

Table 1: Asymptotic performances of MPCC codes of rate 1/2 with rate k/n constituent codes.

code	modified serial	parallel
$R = 3/4 \quad m = 2$	0.44 dB	0.44 dB
$R = 3/4 \quad m = 3$	1.07 dB	1.07 dB
$R = 4/5 \quad m = 2$	1.18 dB	1.18 dB
$R = 4/5 \quad m = 3$	1.74 dB	1.74 dB

Tables 2 and 3 deal with MPCC codes with J constituent codes (A, r, s) of rate $\frac{J}{J+1}$. A (A, r, s) coder is a one memory rate $\frac{r+s}{r+s+1}$ recursive convolutional code with r bits included in the recursion and s bits excluded from the recursion. As shown on figure 6, the Tanner graph of these convolutional codes is a tree.

These concatenated codes are anisotropic of degree 2. For modified serial and parallel decoder, the poor performances of the (A, r, s) are balanced with the number $J-1$ of extrinsic information involved

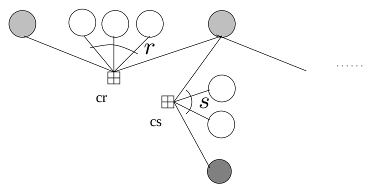

Figure 6: Tanner graph of a (A, r, s) coder.

in the calculation of $EXTR^{(m)}$.

For exemple, the threshold of 0.96dB for the $(A, 3, 1)$ code is close to the one calculated by El Gamal for the rate 1/2 Turbo codes composed of 2 (15, 13) punctured codes. It is interesting to note that for different multiple PCC there is no threshold E_b/N_{0S}. For exemple, let's consider the case of the MPCC code with 3 constituent codes $(A, 1, 2)$. The two different functions f_{E_b/N_0} and the mean of these functions are given on figure 7. Due to the bad protection of the s bits excluded from the recursion, for all values E_b/N_0, the mean function (dashed line) always cross the straight line $SNR_{EXTR} = \frac{1}{2} \times SNR_{APRI}$.

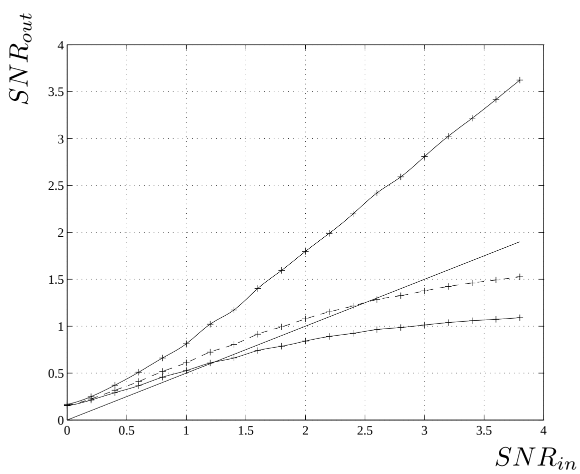

Figure 7: Functions f_{E_b/N_0} for the MPCC code with 3 constituent codes $(A, 1, 2)$.

We have compared the thresholds obtained with gaussian approximation and the performances of the serial, modified serial (dashed line) and parallel decoder for a multiple PCC code composed of

Table 2: Asymptotic performances of multiple PCC codes of rate 1/2 with 3 rate $\frac{3}{4}$ one memory convolutional code rate.

code	serial	modified serial	parallel
$(A, 3, 0)$	3.5dB	1.22 dB	1.22dB
$(A, 2, 1)$	-	-	-
$(A, 1, 2)$	-	-	-

Table 3: Asymptotic performances of multiple PCC codes of rate 1/2 with 4 rate $\frac{4}{5}$ one memory convolutional code rate.

code	serial	modified serial	parallel
$(A, 4, 0)$	4.3dB	1.19 dB	1.19 dB
$(A, 3, 1)$	3 dB	0.96 dB	0.96 dB
$(A, 2, 2)$	-	-	-
$(A, 1, 3)$	-	-	-

4 codes $(A, 3, 1)$ and 3 random interleavers of size N=65536 bits on figure 8.

For the modified serial and parallel decoders, the thresholds are slightly pessimistic (about 0.1dB). We can clearly observe two limitations on the performances of the PCC codes : the limitation due to the convergence of the iterative decoder and the limitation called floor effect due to the low weight codewords.

When the size of the interleavers is smaller, the independence condition is no longer applicable. We have a degradation in the performances of the iterative decoder due to the cycles in the graph. Different stategies are possible to reduce this degradation. A first solution is to build the interleaver in order to increase the girth of the graph [7]. Another solution is to use modified parallel decoders called extended parallel decoders in [8].

5 Conclusion

In this paper, we have generalized the gaussian approximation to multiple PCC codes. For long interleaver size, the thresholds calculated with the gaussian approximation can be used to choose the best constituent codes of multiple PCC codes. We have also shown that the threshold for multiple

PCC codes composed of one memory convolutional codes are close to better than the best turbo codes threshold determined in [2].

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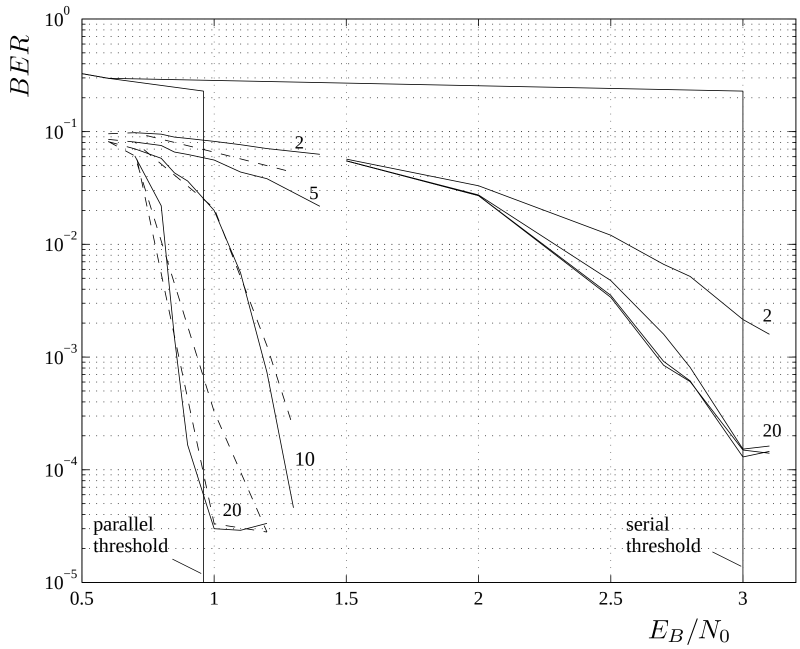

Figure 8: Performances BER versus E_b/N_0 for the serial, modified serial (dashed line) and parallel decoder m=2,5,10 and 20.